

# $S_3 \times \mathbb{Z}_2$ model for neutrino mass matrices

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## Abstract

We propose a model for lepton mass matrices based on the seesaw mechanism, a complex scalar gauge singlet and a horizontal symmetry  $S_3 \times \mathbb{Z}_2$ . In a suitable weak basis, the charged-lepton mass matrix and the neutrino Dirac mass matrix are diagonal, but the vacuum expectation value of the scalar gauge singlet renders the Majorana mass matrix of the right-handed neutrinos non-diagonal, thereby generating lepton mixing. When the symmetry  $S_3$  is not broken in the scalar potential, the effective light-neutrino Majorana mass matrix enjoys  $\mu$ - $\tau$  interchange symmetry, thus predicting maximal atmospheric neutrino mixing together with  $U_{e3} = 0$ . A partial and less predictive form of  $\mu$ - $\tau$  interchange symmetry is obtained when the symmetry  $S_3$  is softly broken in the scalar potential. Enlarging the symmetry group  $S_3 \times \mathbb{Z}_2$  by an additional discrete electron-number symmetry  $\mathbb{Z}_2^{(e)}$ , a more predictive model is obtained, which is in practice indistinguishable from a previous one based on the group  $D_4$ .

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**Introduction** The very precise data now existing on neutrino mass-squared differences and on lepton mixing [1], and the prospects of rapid experimental developments in this field, invite theorists to construct models for the lepton mass matrices, in an effort to exploit and to understand the symmetries and hierarchies suggested by the data. Among them, most prominent are the possible maximality of the atmospheric neutrino mixing angle  $\theta_{23}$  and the smallness of the mixing-matrix element  $U_{e3}$ . Together, they suggest the existence of a  $\mu$ - $\tau$  interchange symmetry in the (effective) light-neutrino Majorana mass matrix  $\mathcal{M}_\nu$ , taken in the basis where the charged-lepton mass matrix is diagonal [2]. Such a symmetry, embodied in

$$\mathcal{M}_\nu = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}, \quad (1)$$

automatically leads to, simultaneously,  $U_{e3} = 0$  and  $\theta_{23} = \pi/4$ . Various authors have dwelt on the matrix (1), and on generalizations thereof, in the past [3, 4, 5, 6]; we, in particular, have shown that it may be obtained either from a model based on family lepton-number symmetries [7] or from a model based on the discrete eight-element group  $D_4$  [8].

We show in this letter that the matrix (1) may also be obtained from a model based on the smaller discrete group  $S_3$ , a group which has a long tradition in model building [9]. The model presented here also suggests a generalization of the matrix (1), wherein

$$\mathcal{M}_\nu^{-1} = \begin{pmatrix} r & s & s \\ s & pe^{i\psi} & q \\ s & q & pe^{-i\psi} \end{pmatrix}. \quad (2)$$

This generalization, while leading to neither  $U_{e3} = 0$  nor  $\theta_{23} = \pi/4$ , seems interesting in itself.

We note that, in [4], the mass matrix (1) has been generalized in such a way that  $\theta_{23}$  differs from  $\pi/4$ , while  $U_{e3} = 0$  remains intact; on the other hand, with matrix (2)—as we shall see later—the deviation of  $U_{e3}$  from zero is correlated with the deviation of  $\theta_{23}$  from  $\pi/4$ .

**The model** We work in the context of a non-supersymmetric  $SU(2)_L \times U(1)$  framework. The three left-handed lepton  $SU(2)_L$  doublets are denoted by  $D_{e,\mu,\tau}$ . The three right-handed charged-lepton  $SU(2)_L$  singlets are  $e_R$ ,  $\mu_R$  and  $\tau_R$ . We further introduce three  $SU(2)_L$  singlet right-handed neutrinos  $\nu_{eR}$ ,  $\nu_{\mu R}$  and  $\nu_{\tau R}$ , in order to enable the seesaw mechanism for suppressing the light-neutrino masses [10]. In our model there are three Higgs  $SU(2)_L$  doublets  $\phi_{1,2,3}$ . In exact analogy to the  $D_4$  model [8], we introduce a symmetry

$$\mathbb{Z}_2^{(\text{aux})} : \quad \nu_{eR}, \nu_{\mu R}, \nu_{\tau R}, \phi_1, e_R \text{ change sign.} \quad (3)$$

Instead of *two real* neutral scalar  $SU(2)_L$  singlets, as in [8], we use *one complex* neutral scalar  $SU(2)_L$  singlet,  $\chi$ . Again in exact analogy to [8], we define a symmetry

$$\mathbb{Z}_2^{(\text{tr})} : \quad D_\mu \leftrightarrow D_\tau, \mu_R \leftrightarrow \tau_R, \nu_{\mu R} \leftrightarrow \nu_{\tau R}, \chi \rightarrow \chi^*, \phi_3 \rightarrow -\phi_3. \quad (4)$$

This  $\mathbb{Z}_2^{(\text{tr})}$  is the  $\mu$ - $\tau$  interchange symmetry. The crucial difference between the  $D_4$  model [8] and the present  $S_3$  one is that, while in the  $D_4$  model there was a symmetry  $\mathbb{Z}_2^{(\tau)}$  which, together with  $\mathbb{Z}_2^{(\text{tr})}$ , generated a group  $D_4$ , in the  $S_3$  model we introduce instead a symmetry  $\mathbb{Z}_3$  which, together with  $\mathbb{Z}_2^{(\text{tr})}$ , generates a group  $S_3$  [11]. With  $\omega \equiv \exp(2i\pi/3)$ , we impose

$$\mathbb{Z}_3 : \begin{aligned} D_\mu &\rightarrow \omega D_\mu, & D_\tau &\rightarrow \omega^2 D_\tau, \\ \mu_R &\rightarrow \omega \mu_R, & \tau_R &\rightarrow \omega^2 \tau_R, \\ \nu_{\mu R} &\rightarrow \omega \nu_{\mu R}, & \nu_{\tau R} &\rightarrow \omega^2 \nu_{\tau R}, \\ \chi &\rightarrow \omega \chi, & \chi^* &\rightarrow \omega^2 \chi^*. \end{aligned} \quad (5)$$

Thus,  $(D_\mu, D_\tau)$ ,  $(\mu_R, \tau_R)$ ,  $(\nu_{\mu R}, \nu_{\tau R})$  and  $(\chi, \chi^*)$  are doublets of  $S_3$ . The Higgs  $SU(2)_L$  doublet  $\phi_3$  changes sign under the odd permutations of  $S_3$ , but stays invariant under the cyclic permutations.

The Yukawa Lagrangian symmetric under  $S_3 \times \mathbb{Z}_2^{(\text{aux})}$  is

$$\begin{aligned} \mathcal{L}_Y = & - \left[ y_1 \bar{D}_e \nu_{eR} + y_2 \left( \bar{D}_\mu \nu_{\mu R} + \bar{D}_\tau \nu_{\tau R} \right) \right] \tilde{\phi}_1 \\ & - y_3 \bar{D}_e e_R \phi_1 - y_4 \left( \bar{D}_\mu \mu_R + \bar{D}_\tau \tau_R \right) \phi_2 - y_5 \left( \bar{D}_\mu \mu_R - \bar{D}_\tau \tau_R \right) \phi_3 \\ & + y_\chi^* \nu_{eR}^T C^{-1} (\nu_{\mu R} \chi^* + \nu_{\tau R} \chi) \\ & + \frac{z_\chi^*}{2} \left( \nu_{\mu R}^T C^{-1} \nu_{\mu R} \chi + \nu_{\tau R}^T C^{-1} \nu_{\tau R} \chi^* \right) + \text{H.c.}, \end{aligned} \quad (6)$$

where  $\tilde{\phi}_1 \equiv i\tau_2 \phi_1^*$ . There is also an  $S_3 \times \mathbb{Z}_2^{(\text{aux})}$ -invariant Majorana mass term

$$\mathcal{L}_M = \frac{m^*}{2} \nu_{eR}^T C^{-1} \nu_{eR} + m'^* \nu_{\mu R}^T C^{-1} \nu_{\tau R} + \text{H.c.} \quad (7)$$

The second term in the right-hand side of (7) differs, in a crucial fashion, from the analogous term in the  $D_4$  model—see equation (9) of [8].

With vacuum expectation values (VEVs)  $\langle 0 | \phi_j^0 | 0 \rangle = v_j$  for  $j = 1, 2, 3$ , one obtains

$$\begin{aligned} m_e &= |y_3 v_1|, \\ m_\mu &= |y_4 v_2 + y_5 v_3|, \\ m_\tau &= |y_4 v_2 - y_5 v_3|. \end{aligned} \quad (8)$$

The  $\mu$ - $\tau$  interchange symmetry  $\mathbb{Z}_2^{(\text{tr})}$  is spontaneously broken by the VEV of  $\phi_3^0$ , so that the  $\mu$  and  $\tau$  charged leptons acquire different masses. The smallness of  $m_\mu$  relative to  $m_\tau$  may be explained by requiring the model to be invariant under an additional, softly broken symmetry [12].

The neutrino Dirac mass matrix is

$$M_D = \text{diag}(a, b, b), \text{ with } a = y_1^* v_1, \ b = y_2^* v_1. \quad (9)$$

When the singlet  $\chi$  acquires a VEV  $\langle 0 | \chi | 0 \rangle = W$ , one obtains Majorana mass terms for the right-handed neutrinos:

$$\mathcal{L}_{M_R} = -\frac{1}{2} (\bar{\nu}_{eR}, \bar{\nu}_{\mu R}, \bar{\nu}_{\tau R}) M_R C \begin{pmatrix} \bar{\nu}_{eR}^T \\ \bar{\nu}_{\mu R}^T \\ \bar{\nu}_{\tau R}^T \end{pmatrix} + \text{H.c.}, \quad (10)$$

with

$$M_R = \begin{pmatrix} m & y_\chi W & y_\chi W^* \\ y_\chi W & z_\chi W^* & m' \\ y_\chi W^* & m' & z_\chi W \end{pmatrix}. \quad (11)$$

We next perform a rephasing of the fields,

$$\begin{aligned} \nu_{\mu R} &\rightarrow e^{i\alpha} \nu_{\mu R}, & D_\mu &\rightarrow e^{i\alpha} D_\mu, & \mu_R &\rightarrow e^{i\alpha} \mu_R, \\ \nu_{\tau R} &\rightarrow e^{-i\alpha} \nu_{\tau R}, & D_\tau &\rightarrow e^{-i\alpha} D_\tau, & \tau_R &\rightarrow e^{-i\alpha} \tau_R, \end{aligned} \quad \text{with } \alpha \equiv \arg W, \quad (12)$$

to obtain

$$M_R = \begin{pmatrix} m & y_\chi |W| & y_\chi |W| \\ y_\chi |W| & z_\chi |W| e^{-3i\alpha} & m' \\ y_\chi |W| & m' & z_\chi |W| e^{3i\alpha} \end{pmatrix}. \quad (13)$$

We see that the matrix  $M_R$  has become  $\mu$ - $\tau$  symmetric after the rephasing, provided  $W^3$  is real ( $e^{3i\alpha} = \pm 1$ ). We shall see shortly that it is indeed possible to enforce this. If  $M_R$  is  $\mu$ - $\tau$  symmetric, and since  $M_D$  also enjoys the  $\mu$ - $\tau$  interchange symmetry, it follows, by applying the seesaw mechanism,<sup>1</sup> that

$$\mathcal{M}_\nu = -M_D^T M_R^{-1} M_D \quad (14)$$

is  $\mu$ - $\tau$  symmetric, i.e. it is of the form (1).

We thus find that it is possible to produce a neutrino mass matrix of the form (1), which leads to  $U_{e3} = 0$  and  $\theta_{23} = \pi/4$ , out of a model with symmetry  $S_3 \times \mathbb{Z}_2^{(\text{aux})}$  with three Higgs  $SU(2)_L$  doublets—two of which are  $S_3$ -invariant while the third one changes sign under the odd permutations of  $S_3$ . The charged-lepton mass matrix is automatically diagonal, hence there are no flavour-changing neutral currents *at tree level* in the charged-lepton sector—such interactions appear, though, already at the one-loop level [13].

**The scalar potential** Because of the symmetry  $S_3 \times \mathbb{Z}_2^{(\text{aux})}$ , the scalar potential is

$$\begin{aligned} V = & \mu_\chi |\chi|^2 + \lambda |\chi|^4 + \sum_{j=1}^3 \left( \phi_j^\dagger \phi_j \right) \left( \mu_j + a_j \phi_j^\dagger \phi_j + b_j |\chi|^2 \right) \\ & + \sum_{j < k} \left[ a_{jk} \left( \phi_j^\dagger \phi_j \right) \left( \phi_k^\dagger \phi_k \right) + b_{jk} \left( \phi_j^\dagger \phi_k \right) \left( \phi_k^\dagger \phi_j \right) + c_{jk} \left( \phi_j^\dagger \phi_k \right)^2 + c_{jk}^* \left( \phi_k^\dagger \phi_j \right)^2 \right] \\ & + m_\chi \left( \chi^3 + \chi^{*3} \right). \end{aligned} \quad (15)$$

Only the term in the last line of (15) feels the phase of  $\chi$ . If its coefficient  $m_\chi$  is negative, then the phase of the VEV  $W$  will adjust so that  $W^3$  is real and positive, i.e.  $\alpha$  will be either 0 or  $\pm 2\pi/3$ ; if  $m_\chi$  is positive, then  $\alpha$  will be either  $\pi$  or  $\pm \pi/3$ , in order that  $W^3$  is real and negative. In any case,  $W^3$  is real. This is precisely what is needed in order to obtain a  $\mu$ - $\tau$ -symmetric  $\mathcal{M}_\nu$ .

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<sup>1</sup>We assume that  $m$ ,  $m'$  and the VEV  $W$  are all of the same very large order of magnitude.

The situation is modified if we allow the symmetry  $\mathbb{Z}_3$  of (5) to be softly broken by terms of dimension one, or one and two, while keeping both  $\mathbb{Z}_2^{(\text{aux})}$  and  $\mathbb{Z}_2^{(\text{tr})}$  unscathed.<sup>2</sup> There are only two such terms, namely

$$\mu'_\chi (\chi^2 + \chi^{*2}) + M (\chi + \chi^*), \quad (16)$$

with real constants  $\mu'_\chi$ ,  $M$ . These terms get added to  $V$  in (15), which does not change otherwise. The phase of  $W$  becomes arbitrary. The matrix  $M_R$  in (13) does not respect  $\mu$ - $\tau$  interchange symmetry any more, rather only a partial version thereof.

If one worries about cosmological domain walls, then one may want to eliminate from the Lagrangian all exact discrete symmetries. This one may do by breaking  $\mathbb{Z}_2^{(\text{aux})}$  and  $\mathbb{Z}_2^{(\text{tr})}$ , together with  $\mathbb{Z}_3$ , softly by terms of dimension two. This amounts to the addition, to the scalar potential (15), of all terms  $\phi_j^\dagger \phi_k$  with  $j \neq k$ . (When  $\mathbb{Z}_2^{(\text{tr})}$  is broken softly, the terms of (16) also have to be generalized to  $\mu'_\chi \chi^2 + M\chi + \text{H.c.}$  with complex  $\mu'_\chi$ ,  $M$ .) However, this soft breaking only affects the values of the  $v_j$  and has no influence on the lepton mass matrices. It is thus irrelevant for the following discussion.

**Reproducing the  $D_4$  model** In the  $D_4$  model [8] there is an *accidental* symmetry

$$\mathbb{Z}_2^{(e)} : D_e, e_R, \nu_{eR}, \chi \text{ change sign.} \quad (17)$$

In the context of the present  $S_3$  model, one may promote that symmetry to *fundamental* and impose it on the Lagrangian from the start. It enforces  $z_\chi = 0$ , hence  $(M_R)_{22} = (M_R)_{33} = 0$ ; since  $M_R = -M_D \mathcal{M}_\nu^{-1} M_D^T$  and  $M_D$  is diagonal, this means the vanishing of the  $(\mu, \mu)$  and  $(\tau, \tau)$  matrix elements of  $\mathcal{M}_\nu^{-1}$ . The phase  $\alpha$  of  $W$  is irrelevant when  $(M_R)_{22} = (M_R)_{33} = 0$ , since it may be rephased away as in (12), and the model is automatically  $\mu$ - $\tau$  symmetric. Thus, in the  $S_3$  model with the extra  $\mathbb{Z}_2^{(e)}$  symmetry one has

$$\mathcal{M}_\nu^{-1} = \begin{pmatrix} r & s & s \\ s & 0 & q \\ s & q & 0 \end{pmatrix}, \quad (18)$$

i.e.  $(\mathcal{M}_\nu^{-1})_{\mu\mu} = (\mathcal{M}_\nu^{-1})_{\tau\tau} = 0$ .

On the other hand, in the  $D_4$  model [8] one has  $(\mathcal{M}_\nu^{-1})_{\mu\tau} = 0$  but  $(\mathcal{M}_\nu^{-1})_{\mu\mu} = (\mathcal{M}_\nu^{-1})_{\tau\tau} \neq 0$ . We shall demonstrate now that the matrix (18) is equivalent to the mass matrix of the  $D_4$  model.

In general,  $\mathcal{M}_\nu$  is diagonalized as

$$U^T \mathcal{M}_\nu U = \text{diag} (m_1, m_2, m_3), \quad (19)$$

where  $U$  is the lepton mixing matrix and the  $m_j$  are the (real and non-negative) neutrino masses. Equivalently,

$$\mathcal{M}_\nu^{-1} = U \text{diag} (m_1^{-1}, m_2^{-1}, m_3^{-1}) U^T. \quad (20)$$

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<sup>2</sup>If we also allow the soft breaking of (5) by terms of dimension three, then Majorana mass terms  $\nu_{eR}^T C^{-1} (\nu_{\mu R} + \nu_{\tau R})$  and  $\nu_{\mu R}^T C^{-1} \nu_{\mu R} + \nu_{\tau R}^T C^{-1} \nu_{\tau R}$  are also present in the Lagrangian and, after  $\chi$  gets a VEV, the  $\mu$ - $\tau$  interchange symmetry is destroyed altogether.

The unitary  $U$  is parametrized as

$$\begin{aligned}
U &= \text{diag}(e^{i\vartheta_1}, e^{i\vartheta_2}, e^{i\vartheta_3}) \\
&\times \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ -s_{23}s_{12} + c_{23}s_{13}c_{12}e^{i\delta} & s_{23}c_{12} + c_{23}s_{13}s_{12}e^{i\delta} & -c_{23}c_{13} \end{pmatrix} \\
&\times \text{diag}(e^{i\Delta/2}, 1, e^{i\Omega/2}),
\end{aligned} \tag{21}$$

where  $c_{ij} \equiv \cos \theta_{ij}$ ,  $s_{ij} \equiv \sin \theta_{ij}$  and  $\theta_{12}$  is the solar mixing angle. The phases  $\vartheta_i$  are unphysical; physical are only the Dirac phase  $\delta$  and the Majorana phases  $\Delta$  and  $\Omega$ . In the case of  $\mu$ - $\tau$  symmetric  $\mathcal{M}_\nu^{-1}$ , one has—see for instance [14]— $\vartheta_2 = \vartheta_3$ ,  $\theta_{13} = 0$  and  $\theta_{23} = \pi/4$ ; the vanishing of  $\theta_{13}$  allows one to write the non-diagonal matrix on the right-hand side of (21) as a product  $U_{23}U_{12}$ , where  $U_{23}$  and  $U_{12}$  are responsible for the mixing in the atmospheric and solar neutrino sector, respectively.

Now we turn to the matrix (18), which we can transform according to

$$T\mathcal{M}_\nu^{-1}T = \begin{pmatrix} r & s & s \\ s & q & 0 \\ s & 0 & q \end{pmatrix}, \quad \text{with} \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & u^* \\ 0 & u^* & u \end{pmatrix} \quad \text{and} \quad u = \frac{e^{i\pi/4}}{\sqrt{2}}. \tag{22}$$

Thus,  $T\mathcal{M}_\nu^{-1}T$  has precisely the form of the mass matrix of the  $D_4$  model, and from (20) it is clear that it can be diagonalized by a  $U$  appropriate for a  $\mu$ - $\tau$  symmetric matrix. Since  $\vartheta_2 = \vartheta_3$ , in the diagonalization of  $T\mathcal{M}_\nu^{-1}T$  the product  $TU_{23}$  occurs. Now,

$$TU_{23} = U_{23} \text{diag}(1, 1, i), \quad \text{where} \quad U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & \rho \\ 0 & \rho & -\rho \end{pmatrix} \quad \text{and} \quad \rho = \frac{1}{\sqrt{2}}. \tag{23}$$

The matrix  $U_{12}$  commutes with  $\text{diag}(1, 1, i)$ . Therefore, the difference between the  $D_4$  model and the  $S_3 \times \mathbb{Z}_2^{(\text{aux})} \times \mathbb{Z}_2^{(\text{e})}$  model amounts to the modification  $\Omega \rightarrow \Omega + \pi$ . Since  $\Omega$  is a free parameter, the two models are in practice equivalent.

Just as in the  $D_4$  model [8], the zero in the matrix (18), or—equivalently—the zero in  $T\mathcal{M}_\nu^{-1}T$  of (22), leads to

$$\frac{s_{12}^2 e^{i\Delta}}{m_1} + \frac{c_{12}^2}{m_2} + \frac{e^{i\Omega}}{m_3} = 0. \tag{24}$$

As we have shown in [8], the constraint (24) implies, given the known experimental values, a normal mass spectrum  $m_1 < m_2 < m_3$ , with  $m_1$  either in the range  $3$  to  $9 \times 10^{-3}$  eV, or larger than  $14 \times 10^{-3}$  eV; these numbers hold for the best-fit values of the mass-squared differences as given in [1]. A further prediction is  $|\langle m \rangle| = m_1 m_2 / m_3$ , where  $|\langle m \rangle|$  is the effective mass relevant for neutrinoless  $\beta\beta$  decay.

**Generalization of the  $\mu$ - $\tau$  interchange symmetry** We now abandon the symmetry  $\mathbb{Z}_2^{(\text{e})}$  and return to the general  $M_R$  of the  $S_3$  model, given in (13). Since  $\mathcal{M}_\nu^{-1} = -M_D^{-1}M_R M_D^{T-1}$ , and since  $M_D = \text{diag}(a, b, b)$  is diagonal, one obtains  $\mathcal{M}_\nu^{-1}$  of the form (2).

In general, the symmetric matrix  $\mathcal{M}_\nu^{-1}$  contains nine parameters: the six moduli of its matrix elements and three rephasing-invariant phases, because one may independently rephase the three left-handed neutrinos, thereby eliminating three phases in  $\mathcal{M}_\nu$ , or equivalently in  $\mathcal{M}_\nu^{-1}$ . To those nine parameters correspond nine observables: the three neutrino masses  $m_{1,2,3}$ , the three mixing angles  $\theta_{12,13,23}$ , the Dirac phase  $\delta$  and the Majorana phases  $\Delta$  and  $\Omega$ .

In the case of (full)  $\mu$ - $\tau$  interchange symmetry, i.e. when  $e^{i\psi} = \pm 1$  in (2), three observables are predicted:  $\theta_{23} = \pi/4$ ,  $\theta_{13} = 0$  and the Dirac phase is meaningless because  $\theta_{13} = 0$ . To those three predicted observables correspond three rephasing-invariant relations among the parameters of  $\mathcal{M}_\nu$ , or of  $\mathcal{M}_\nu^{-1}$ :

$$\left| (\mathcal{M}_\nu^{-1})_{e\mu} \right| = \left| (\mathcal{M}_\nu^{-1})_{e\tau} \right|, \quad (25)$$

$$\left| (\mathcal{M}_\nu^{-1})_{\mu\mu} \right| = \left| (\mathcal{M}_\nu^{-1})_{\tau\tau} \right|, \quad (26)$$

$$\arg \left\{ \left[ (\mathcal{M}_\nu^{-1})_{e\mu} \right]^2 (\mathcal{M}_\nu^{-1})_{\tau\tau} \right\} = \arg \left\{ \left[ (\mathcal{M}_\nu^{-1})_{e\tau} \right]^2 (\mathcal{M}_\nu^{-1})_{\mu\mu} \right\}. \quad (27)$$

In the case of the matrix (2), the condition (27) does not apply. One has an *incomplete*  $\mu$ - $\tau$  interchange symmetry, wherein conditions (25) and (26) apply, but not condition (27). The matrix (2) has seven real physical parameters. As we will see,  $\psi \neq 0$  leads both  $\cos 2\theta_{23}$  and  $s_{13}$  to be non-zero, and in general there will also be a non-zero Dirac phase  $\delta$ . Since the matrix (2) has only one parameter more than matrix (1), it must predict two relations: two of the observables  $\cos 2\theta_{23}$ ,  $s_{13}$  and  $\delta$  must be functions of the third one and of the remaining observables, which are  $m_{1,2,3}$ ,  $\theta_{12}$ ,  $\Delta$  and  $\Omega$ . Since the Majorana phases are hardly accessible by experiment, our aim is to derive observable consequences of incomplete  $\mu$ - $\tau$  interchange symmetry which do not involve those phases.

It is convenient to use the facts that, from experiment, it is known that the atmospheric mixing angle  $\theta_{23}$  is close to  $\pi/4$  and that  $\theta_{13}$  is small. Thus we define the parameters

$$\nu \equiv \cos 2\theta_{23}, \quad (28)$$

$$\epsilon \equiv s_{13} e^{i\delta}, \quad (29)$$

the latter of which is complex. Experimentally [1],  $|\nu| < 0.28$  at 90% confidence level and  $|\epsilon| < 0.22$  at  $3\sigma$  level. In the case of full  $\mu$ - $\tau$  interchange symmetry,  $\nu = \epsilon = 0$  and there are no restrictions on all other observables, i.e. on  $m_{1,2,3}$ ,  $\theta_{12}$ ,  $\Delta$  and  $\Omega$ . In the case of incomplete  $\mu$ - $\tau$  interchange symmetry,  $\nu$  and  $\epsilon$  in general do not vanish and, when they are non-zero, some restrictions may apply on the other observables.

Adding (25) and (26) and using (20), one finds that

$$0 = \left| (\mathcal{M}_\nu^{-1})_{e\mu} \right|^2 + \left| (\mathcal{M}_\nu^{-1})_{\mu\mu} \right|^2 - \left| (\mathcal{M}_\nu^{-1})_{e\tau} \right|^2 - \left| (\mathcal{M}_\nu^{-1})_{\tau\tau} \right|^2 \quad (30)$$

$$= (\mathcal{M}_\nu^{-1} \mathcal{M}_\nu^{-1*})_{\mu\mu} - (\mathcal{M}_\nu^{-1} \mathcal{M}_\nu^{-1*})_{\tau\tau} \quad (31)$$

$$= \sum_{j=1}^3 \frac{|U_{\mu j}|^2 - |U_{\tau j}|^2}{m_j^2} \quad (32)$$

$$\begin{aligned}
&= \left( c_{23}^2 - s_{23}^2 \right) \left( \frac{s_{12}^2 - s_{13}^2 c_{12}^2}{m_1^2} + \frac{c_{12}^2 - s_{13}^2 s_{12}^2}{m_2^2} - \frac{c_{13}^2}{m_3^2} \right) \\
&\quad + 4 \left( \frac{1}{m_1^2} - \frac{1}{m_2^2} \right) c_{23} s_{23} s_{13} c_{12} s_{12} \cos \delta.
\end{aligned} \tag{33}$$

This condition is particularly useful since it does not involve the Majorana phases. It translates into

$$\left( \frac{s_{12}^2 - |\epsilon|^2 c_{12}^2}{m_1^2} + \frac{c_{12}^2 - |\epsilon|^2 s_{12}^2}{m_2^2} + \frac{|\epsilon|^2 - 1}{m_3^2} \right) \nu + 2 \left( \frac{1}{m_1^2} - \frac{1}{m_2^2} \right) c_{12} s_{12} \sqrt{1 - \nu^2} \operatorname{Re} \epsilon = 0. \tag{34}$$

Numerically, we shall use (34) to determine  $\nu$  as a function of  $\epsilon$ , for various values of the neutrino masses and of the mixing angle  $\theta_{12}$ . Since  $|\epsilon|^2$  and  $\nu^2$  are in any case rather small, (34) is an almost linear relationship between  $\nu$  and  $\operatorname{Re} \epsilon$ .

We next consider the constraint (25) by itself alone. It is equivalent to the existence of a phase  $\varphi$  such that

$$0 = \left( \mathcal{M}_\nu^{-1} \right)_{e\mu} - e^{i\varphi} \left( \mathcal{M}_\nu^{-1} \right)_{e\tau} \tag{35}$$

$$= \sum_{j=1}^3 \frac{U_{ej} (U_{\mu j} - e^{i\varphi} U_{\tau j})}{m_j}. \tag{36}$$

This is the equation of a triangle in the complex plane—it states that the sum of three complex numbers vanishes, i.e. that those three numbers form a triangle in the complex plane. The triangle (36) involves the Majorana phases. It is convenient to remove those phases, since they are in practice very difficult to observe experimentally. One does this by considering the inequality, which follows from (36),<sup>3</sup>

$$\sum_{j=1}^3 \frac{|U_{ej} (U_{\mu j} - e^{i\varphi} U_{\tau j})|^4}{m_j^4} - 2 \sum_{j < k} \frac{|U_{ej} (U_{\mu j} - e^{i\varphi} U_{\tau j})|^2 |U_{ek} (U_{\mu k} - e^{i\varphi} U_{\tau k})|^2}{m_j^2 m_k^2} \leq 0. \tag{37}$$

Notice that, using (21), one has

$$\begin{aligned}
\left| U_{e1} (U_{\mu 1} - e^{i\varphi} U_{\tau 1}) \right|^2 / c_{13}^2 &= c_{12}^2 s_{12}^2 (1 - \sqrt{1 - \nu^2} \cos \phi) \\
&\quad + c_{12}^4 |\epsilon|^2 (1 + \sqrt{1 - \nu^2} \cos \phi) \\
&\quad + 2 c_{12}^3 s_{12} (\nu \operatorname{Re} \epsilon \cos \phi - \operatorname{Im} \epsilon \sin \phi), \\
\left| U_{e2} (U_{\mu 2} - e^{i\varphi} U_{\tau 2}) \right|^2 / c_{13}^2 &= c_{12}^2 s_{12}^2 (1 - \sqrt{1 - \nu^2} \cos \phi)
\end{aligned} \tag{38}$$

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<sup>3</sup>It is possible to construct a triangle with sides of (real, non-negative) lengths  $a$ ,  $b$  and  $c$  if and only if

$$a \leq b + c, \quad b \leq a + c \quad \text{and} \quad c \leq a + b.$$

It is easily shown that this set of three inequalities is equivalent to the sole inequality [15]

$$a^4 + b^4 + c^4 - 2(a^2 b^2 + a^2 c^2 + b^2 c^2) \leq 0.$$



$$+s_{12}^4 |\epsilon|^2 \left(1 + \sqrt{1 - \nu^2} \cos \phi\right) - 2c_{12}s_{12}^3 (\nu \operatorname{Re} \epsilon \cos \phi - \operatorname{Im} \epsilon \sin \phi), \quad (39)$$

$$\left|U_{e3} \left(U_{\mu 3} - e^{i\varphi} U_{\tau 3}\right)\right|^2 / c_{13}^2 = |\epsilon|^2 \left(1 + \sqrt{1 - \nu^2} \cos \phi\right), \quad (40)$$

where  $\phi \equiv \varphi + \vartheta_3 - \vartheta_2$ .

Numerically, we use (34) to determine  $\nu$  as a function of  $\epsilon$ , for various values of the neutrino masses and of the mixing angle  $\theta_{12}$ . Afterwards, we check whether there is any phase  $\varphi$  for which the inequality (37) is satisfied. If there is, then those values of the neutrino masses, mixing angles and Dirac phase are compatible with incomplete  $\mu$ - $\tau$  interchange symmetry; otherwise they are not. For simplicity we keep  $\theta_{12} = 33^\circ$ ,  $m_2^2 - m_1^2 = 8.1 \times 10^{-5} \text{ eV}^2$  and  $|m_3^2 - m_1^2| = 2.2 \times 10^{-3} \text{ eV}^2$  fixed at their best-fit values [1]. It is important to remark that (38)–(40), just as (34), are symmetric under  $\nu \rightarrow -\nu$ ,  $\operatorname{Re} \epsilon \rightarrow -\operatorname{Re} \epsilon$ . This means that one only has to study the region of positive  $\operatorname{Re} \epsilon$ . Also, (38)–(40) are invariant under  $\operatorname{Im} \epsilon \rightarrow -\operatorname{Im} \epsilon$ ,  $\sin \phi \rightarrow -\sin \phi$ . This means that we only have to consider positive values of  $\operatorname{Im} \epsilon$ , provided we test all possible values of  $\phi$ .

In the case where  $m_3^2 - m_1^2 < 0$ , the situation is rather simple and it is aptly described by fig. 1. The parameter  $\nu$  has the same sign as  $\operatorname{Re} \epsilon$  but it is much smaller in absolute value; the atmospheric mixing angle is, for all practical purposes, maximal; in the limit of very small  $m_3$  the relation  $\theta_{23} = 45^\circ$  becomes exact—see (34) and fig. 1. The exact value of  $\operatorname{Im} \epsilon$  is practically immaterial in the determination of  $\nu$  as a function of  $\operatorname{Re} \epsilon$ . The inequality (37) is always satisfied, hence it has no bearing on the overall picture.

In the case where  $m_3^2 - m_1^2 > 0$  the situation is different. The parameters  $\nu$  and  $\operatorname{Re} \epsilon$  have opposite signs and  $\nu$  is not necessarily small. On the other hand, the determination of  $\nu$  as a function of  $\operatorname{Re} \epsilon$  is, once again, largely insensitive to the exact value of  $\operatorname{Im} \epsilon$ . Typical values are displayed in fig. 2.

When  $m_3^2 - m_1^2 > 0$ , inequality (37) introduces a complication because in this case there are values of the pair  $(\nu, \epsilon)$  for which that inequality is satisfied by no phase  $\varphi$  at all. With  $\nu$  and  $\operatorname{Re} \epsilon$  obeying the relation (34), this happens when  $\operatorname{Im} \epsilon \lesssim 0.01$  and  $m_1 \sim 10^{-2} \text{ eV}$ . For  $\operatorname{Im} \epsilon = 10^{-4}$  and  $5 \times 10^{-3}$ , the corresponding excluded regions in the  $(\nu, \operatorname{Re} \epsilon)$  plane are depicted in fig. 3. That figure should be superimposed on fig. 2 in order to see which curves or which parts of the curves in that figure are excluded and to find out the range of values of  $m_1^2$  for which an excluded region arises.<sup>4</sup> Of the curves depicted in fig. 2, only the small-dashed line, referring to  $m_1 = 10^{-2} \text{ eV}$ , is affected: for  $\operatorname{Im} \epsilon = 10^{-4}$  that line is almost completely excluded, whereas for  $\operatorname{Im} \epsilon = 5 \times 10^{-3}$  it is excluded if  $\operatorname{Re} \epsilon \gtrsim 0.08$ . Explicitly, we have found that, when  $\operatorname{Im} \epsilon$  vanishes, excluded values of  $\nu$  and  $\operatorname{Re} \epsilon$  arise for  $7.80 \times 10^{-3} \text{ eV} < m_1 < 1.28 \times 10^{-2} \text{ eV}$ ; when  $\operatorname{Im} \epsilon = 5 \times 10^{-3}$ , excluded values of  $\operatorname{Re} \epsilon$  arise only for  $m_1$  in between  $8.49 \times 10^{-3} \text{ eV}$  and  $1.16 \times 10^{-2} \text{ eV}$ .

The excluded regions in the  $(\nu, \operatorname{Re} \epsilon)$  plane can be translated into lower bounds for the  $CP$ -violating phase  $\delta$ . These lower bounds are functions of  $m_1$  and of  $s_{13}$ . Taking  $m_1 = 0.01 \text{ eV}$ , i.e.  $m_1$  in the center of the range where excluded regions occur, we numerically find  $\delta \gtrsim 2.44^\circ$  for  $s_{13} = 0.2$ ,  $\delta \gtrsim 3.34^\circ$  for  $s_{13} = 0.1$ ,  $\delta \gtrsim 3.52^\circ$  for  $s_{13} = 0.01$ , and  $\delta \gtrsim 3.83^\circ$  for  $s_{13} = 0.001$ . (We have confined ourselves to  $\delta$  in the first quadrant, i.e. to the real

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<sup>4</sup>In this comparison we use the fact that the curves in fig. 2 are practically independent of the value of  $\operatorname{Im} \epsilon$ , provided  $\operatorname{Im} \epsilon \lesssim 0.1$ .

and imaginary part of  $\epsilon$  being both positive. The bounds on  $\delta$  in the first quadrant get transferred into the other quadrants by using the symmetries  $\text{Re } \epsilon \rightarrow -\text{Re } \epsilon$ ,  $\nu \rightarrow -\nu$  and  $\text{Im } \epsilon \rightarrow -\text{Im } \epsilon$ ,  $\sin \phi \rightarrow -\sin \phi$  referred to earlier.) One sees that the excluded domain is hardly significant in terms of  $\delta$ . For  $s_{13} \gtrsim 0.1$ , this is qualitatively understandable from the fact that for  $\text{Im } \epsilon = s_{13} \sin \delta \gtrsim 0.01$  there is no exclusion region anymore.

**Conclusions** In this paper we have considered an extension of the Standard Model based on the horizontal symmetry group  $S_3 \times \mathbb{Z}_2$ , the seesaw mechanism and a complex scalar gauge singlet  $\chi$ . Though  $S_3$  is a time-honoured symmetry, the new feature here is the use of the complex scalar gauge singlet, with  $(\chi, \chi^*)$  transforming as a two-dimensional irreducible representation of  $S_3$ —see (4) and (5). The gauge multiplets of our extension are those of the Standard Model, supplemented by two additional Higgs doublets, the scalar singlet and three right-handed neutrino singlets for the seesaw mechanism. The horizontal symmetry enforces diagonal charged-lepton and neutrino Dirac mass matrices. Our model has some freedom with regard to the realization of the symmetry  $S_3 \times \mathbb{Z}_2$ , and this freedom affects the Majorana mass matrix of the right-handed neutrinos. In this way, we are able to recover two mass matrices already found in the literature, derived from different horizontal symmetries, and also the mass matrix (2), a generalization thereof; we regard this flexibility as the distinguishing feature of the  $S_3 \times \mathbb{Z}_2$  model. In terms of the inverted light-neutrino mass matrix (2), our results can be described in the following way:

1. Imposing the additional discrete electron number  $\mathbb{Z}_2^{(e)}$  of (17), one obtains  $\psi = 0$  and  $q = 0$ ; one thus recovers a mass matrix originally derived in [8] from a horizontal symmetry group  $D_4$ .
2. Without  $\mathbb{Z}_2^{(e)}$  and with exact  $S_3 \times \mathbb{Z}_2$  symmetry of the Lagrangian, one gets  $\psi = 0$ , i.e. the  $\mu$ - $\tau$  symmetric mass matrix originally obtained in [7] in a framework of softly broken lepton numbers.
3. Breaking  $S_3 \times \mathbb{Z}_2$  softly in the scalar potential, one obtains the matrix (2) without further restrictions.

The first and second realizations have a  $\mu$ - $\tau$  symmetric mass matrix, with the well-known predictions of maximal atmospheric neutrino mixing and vanishing mixing-matrix element  $U_{e3}$ . If, in addition,  $q = 0$  holds, then the model becomes much more predictive: it requires a normal neutrino mass ordering  $m_1 < m_2 < m_3$ , and the effective mass in neutrinoless  $\beta\beta$  decay is a simple function of the neutrino masses alone [8]. Below the seesaw scale, the first and second realizations cannot be distinguished from the models in [8] and [7], respectively.

With  $\psi \neq 0$ , the mass matrix (2) has seven physical parameters and partially breaks the  $\mu$ - $\tau$  interchange symmetry; the matrix of the absolute values of the elements of the *inverted* neutrino mass matrix (2) is still  $\mu$ - $\tau$  symmetric.<sup>5</sup> In contrast to full  $\mu$ - $\tau$  interchange symmetry, this partial symmetry induces non-zero  $\cos 2\theta_{23}$  and  $s_{13}$ , and  $CP$  violation in neutrino mixing via the Dirac phase  $\delta$ . These three quantities are functions

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<sup>5</sup>Any mass matrix with that property can be transformed into (2) by a phase transformation.

of  $\psi$ . Fixing the neutrino masses and the solar mixing angle, there is an almost linear relation (34) between  $\cos 2\theta_{23}$  and  $s_{13} \cos \delta$ , which is not obfuscated by the Majorana phases. This relation differs substantially depending on the type of neutrino mass spectrum: in the inverted case, atmospheric mixing is always maximal for all practical purposes, even when  $s_{13}$  is close to its experimental upper bound; in the normal case, a large  $s_{13} \cos \delta$  is correlated with a large  $\cos 2\theta_{23}$  with an opposite sign.

Finally, as an additional virtue, we mention that, for a normal neutrino mass spectrum, leptogenesis can naturally be accommodated in the present model with the  $\mu$ - $\tau$  symmetric mass matrices [14]; at least for small  $s_{13}$ , the same must hold with partial  $\mu$ - $\tau$  interchange symmetry.

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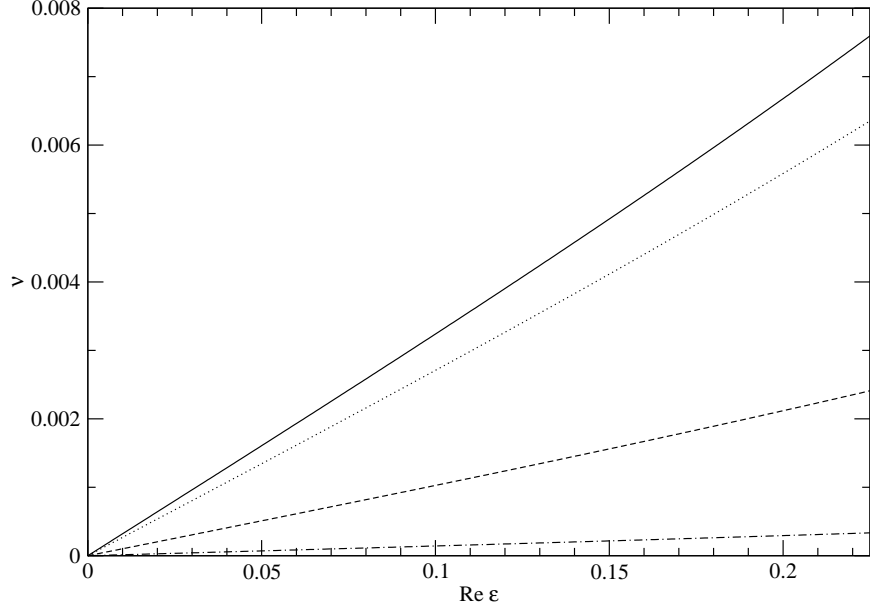


Figure 1:  $\nu$  as a function of  $\text{Re } \epsilon$  in the case  $m_3 < m_{1,2}$ . The figure corresponds to a vanishing  $\text{Im } \epsilon$ , but it would be practically identical for any  $\text{Im } \epsilon$  of order 0.1 or smaller. The full line is for  $m_3^2 = 10^{-1} \text{ eV}^2$ , the dotted line for  $m_3^2 = 10^{-2} \text{ eV}^2$ , the dashed line for  $m_3^2 = 10^{-3} \text{ eV}^2$  and the dashed-dotted line for  $m_3^2 = 10^{-4} \text{ eV}^2$ .

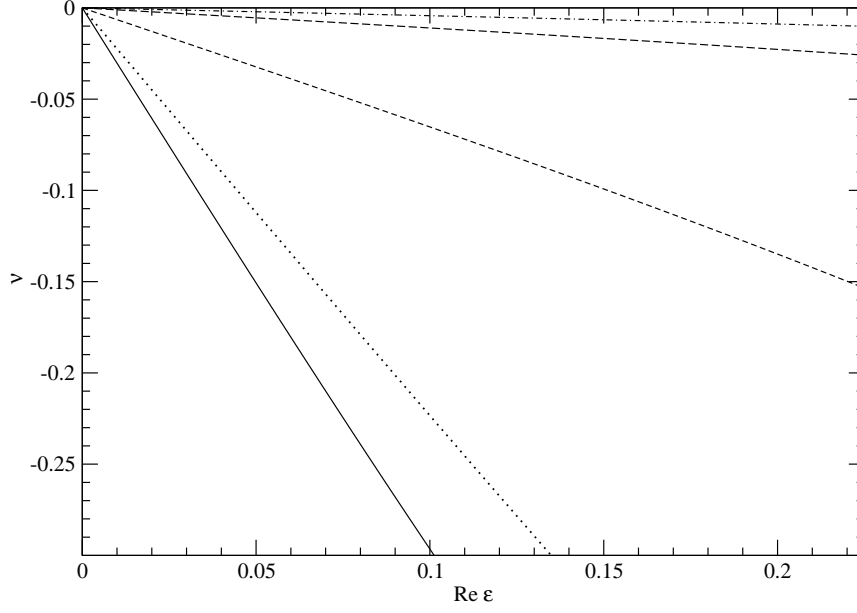


Figure 2:  $\nu$  as a function of  $\text{Re } \epsilon$  in the case  $m_3 > m_{1,2}$  and with  $\text{Im } \epsilon = 0.1$  (a smaller  $\text{Im } \epsilon$  yields practically the same curves, except for the exclusion zones depicted in fig. 3). The full line is for  $m_1^2 = 10^{-6} \text{ eV}^2$ , the dotted line for  $m_1^2 = 10^{-5} \text{ eV}^2$ , the small-dashed line for  $m_1^2 = 10^{-4} \text{ eV}^2$ , the large-dashed line for  $m_1^2 = 10^{-3} \text{ eV}^2$  and the dashed-dotted line for  $m_1^2 = 10^{-2} \text{ eV}^2$ .

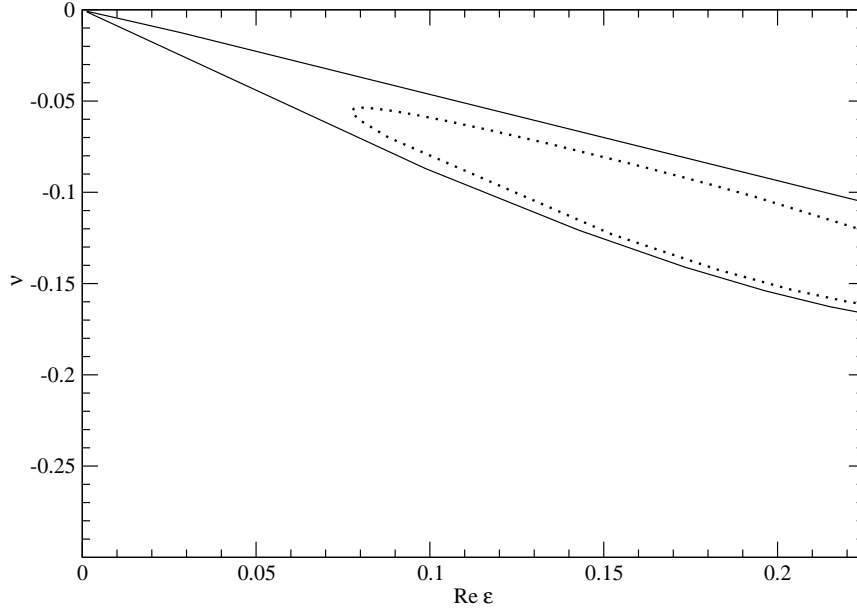


Figure 3: In the case  $m_3 > m_{1,2}$ , the region in the  $\nu$ – $\text{Re } \epsilon$  plane inside the full line is excluded by (37) when  $\text{Im } \epsilon = 10^{-4}$ . The region inside the dotted line is excluded when  $\text{Im } \epsilon = 5 \times 10^{-3}$ . For  $\text{Im } \epsilon$  larger than  $10^{-2}$  there is no excluded region any more.